Generalized min-up/min-down polytopes

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Abstract

Consider a time horizon and a set of possible states for a given system. The system must be in exactly one state at a time. In this paper, we generalize classical results on min-up/min-down constraints for a 2-state system to an n-state system. The minimum-time constraints enforce that if the system switches to state i at time t, then it must remain in state i for a minimum number of time steps. The minimum-time polytope is defined as the convex hull of integer solutions satisfying the minimum-time constraints. A variant of minimum-time constraints is also considered, namely the no-spike constraints. They enforce that if state i is switched on at time t, the system must remain on states $j \geq i$ during a minimum time. Symmetrically, they enforce that if state i is switched off at time t, the system must remain on states $j < i$ during a minimum time. The no-spike polytope is defined as the convex hull of integer solutions satisfying the no-spike constraints. For both the minimum-time polytope and the no-spike polytope, we introduce families of valid inequalities. We prove that these inequalities lead to a complete description of linear size for each polytope.

Consider a time horizon $\mathcal{T} = \{1, ..., T\}$ and a set of possible states $\mathcal{N} = \{0, ..., n\}$ for a given system. The system must be in exactly one state at a time. We introduce *minimum-time constraints* as follows. If the system switches to state $i \in \mathcal{N}$ at time t, then it must remain in state i for at least L^i time steps. The minimum-time polytope is defined as the convex hull of integer solutions satisfying the minimum-time constraints. These constraints directly generalize minimum up and down time constraints from the literature [11, 18] in the sense that the system has an arbitrary number n of possible states, instead of only two states (up and down).

A variant of minimum-time constraints is also considered, namely the *no-spike* constraints. The *no-up-spike* constraints enforce that if state $i \in \mathcal{N}$ is switched on at time $t \in \mathcal{T}$, the system must remain on states $j \geq i$ during at least L^i time steps. This forbids to switch down the states of the system too rapidly after having switched them up. Symmetrically, the *no-down-spike* constraints enforce that if state $i \in \mathcal{N}$ is switched off at time $t \in \mathcal{T}$, the system must remain on states $j < i$ during at least ℓ^i time steps. This forbids to switch up the states of the system too rapidly after having switched them down. The no-spike polytope is defined as the convex hull of integer solutions satisfying the no-spike constraints.

To illustrate, Figure 1 displays an example solution for a 4-state system on 7 time steps satisfying minimum-times $L^i = 3$, $i \in \mathcal{N}$, and Figure 2 displays an example solution for a 4-state system on 10 time steps satisfying no-spike constraints with $L^i = \ell^i = 3$, $i \in \mathcal{N}$.

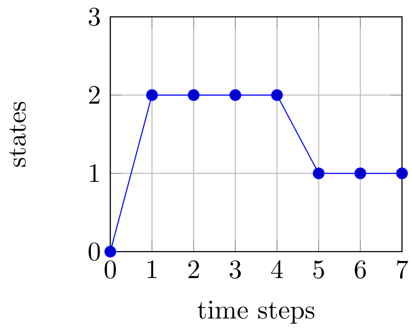


Figure 1: Example of a solution for a system subject to minimum-time constraints

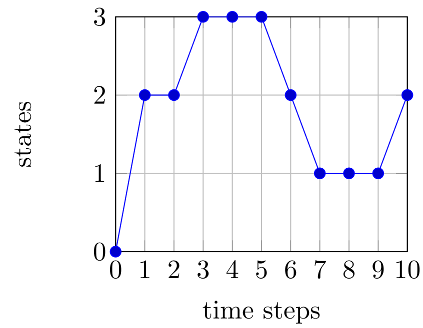


Figure 2: Example of a solution for a system subject to no-spike constraints

Minimum-time and no-spike constraints are of particular interest, because they appear in practical Unit Commitment Problems, typically when production units feature discrete operating states. Minimum-time constraints are very often involved for combined-cycle units, which feature various operating modes, and minimum up and down times on each mode [5, 6, 13]. Moreover, in the French electrical system [8], nuclear units are modeled with discrete production levels and minimum-time constraints, and hydro-electric units are modeled with discrete production levels, and no-spike constraints.

Minimum-time constraints on multiple-state systems also very classical in mixed-integer optimal control problems [1, 9, 3], where optimal policies usually have to avoid very short successive changes. In this context as well, tight MILP formulations for such discrete constraints can prove useful, specifically in decomposed resolution schemes [20].

Related work For a system with only 2 possibles states (up and down, or equivalently 0 and 1), minimum-time constraints and no-spike constraints simplify to min-up/min-down constraints. A complete description of the min-up/min-down polytope is given in [11], in the space of binary variables x indicating on/off status. This description involves an exponential number of inequalities.

The authors of [18] define binary variables u indicating a switch from state 0 to state 1. They introduce turn-on/turn-off inequalities, a polynomial family of inequalities enforcing the min-up/min-down constraints. They prove that along with trivial inequalities, they provide a complete description of the minimum-time polytope in the variable space (x, u). Similar results have been proven independently in [12].

Polytopes associated to couplings of minimum-time constraints with other constraints have also been studied. In [17], the authors propose, for a two-state system, a polyhedral description of the minimum-time constraints coupled to maximum time constraints, enforcing that each state does not remain activated for too long after being switched on. In [2, 19] is proposed a polyhedral study of the coupling between min-up/min-down constraints and knapsack constraints. In [4], the authors propose tight formulations for minimum-time constraints coupled to particular ramping constraints enforcing that transitions are allowed only between consecutive states.

In the context of the Unit Commitment Problem where production units feature continuous production variables, multiple studies propose tight formulations for the coupling between min-up/min-down constraints and production ramping constraints enforcing limited power variations for consecutive time steps [7, 14, 10, 15, 16].

In this paper, we generalize classical results on min-up/min-down constraints for a 2-state system to an n-state system. To this aim, we introduce two new polytopes: the minimum-time polytope and the no-spike polytope. For each polytope, we introduce families of valid inequalities. We prove that these inequalities lead to a complete description of linear size for each polytope.

In Section 1, a formulation for the minimum-time constraints is given, using on/off binary variables x and switch binary variables u. This formulation involves exclusion constraints on variables x, enforcing that exactly one state is used at each time step. In that sense, binary variables x can be seen as *multiple-choice* variables. The formulation also involves extension of turn/on-off inequalities from [18] to n-state systems. The minimum-time polytope is defined in the (x, u) variable space. We introduce a linear-size family of valid inequalities providing a strengthened upper-bound on switch variables u. We prove that the minimum-time polytope can be completely described using these inequalities along with formulation inequalities. In Section 2, a formulation of no-spike constraints is given, also using on/off binary variables x and switch binary variables u. Exclusion constraints are replaced with order constraints on variables x. Therefore, variables x can be seen here as *incremental* variables, leading to an incremental formulation instead of a multiple choice one. We show that the extension of turn/on-off inequalities from [18] involving incremental variables enables to enforce no-spike constraints. The no-spike polytope is accordingly defined in variables space (x, u). Two linear-size families of valid inequalities are introduced: generalized order inequalities and generalized turn-on/off inequalities. We prove that the no-spike polytope can be completely described using these inequalities along with formulation inequalities.

1 The Minimum-Time polytope

In this section, we consider an n-state system with minimum times L^i on each state i of the system.

Using multiple choice variables x_t^i indicating that the system is in state i at time t, the following exclusion constraint holds:

$$\sum_{i \in \mathcal{N}} x_t^i = 1 \qquad \forall t \in \mathcal{T} \tag{1}$$

Additional binary variables u are used to indicate the fact that a state has been switched on at a given time step. Thus, variable u_t^i indicates that state i has been switched on at time t:

$$\begin{cases} u_t^i \geq x_t^i - x_{t-1}^i & \forall t \in \{2, ..., T\},\ \forall i \in \mathcal{N} \quad (2a) \\ u_t^i \leq x_t^i & \forall t \in \mathcal{T},\ \forall i \in \mathcal{N} \quad (2b) \\ u_t^i \leq 1 - x_{t-1}^i & \forall t \in \{2, ..., T\},\ \forall i \in \mathcal{N} \quad (2c) \end{cases}$$

Extending to the n-state case the turn-on constraints introduced for a two-state system in [18], minimum-time constraints can be written as *extended turn-on inequalities*:

$$\sum_{t'=t-L^i+1}^{t} u_{t'}^i \leq x_t^i \qquad \forall t \in \{L^i + 1, ..., T\} \qquad \forall i \in \mathcal{N} \tag{3}$$

The minimum-time polytope is defined as follows:

$$\mathcal{P}^{MT} = conv\Big\{(x, u) \in \{0, 1\}^{(n \times T) \times (n \times T - 1)} \mid (1),\ (2a),\ (2b),\ (2c),\ (3)\Big\}$$

In the following, we introduce a linear-size family of valid inequalities arising from the coupling between (1) and (3). We show that along inequalities (1), (2a), (2b), (2c), (3), this set of inequalities gives a complete linear description of $\mathcal{P}^{MT}$.

1.1 Strengthened Upper Bound (SUB) inequalities

We introduce the Strengthened Upper Bound (SUB) inequalities as follows.

$$u_t^i \leq x_t^i - x_{t-1}^i + \sum_{j \neq i} u_t^j \qquad \forall i \in \mathcal{N},\ \forall t \in \mathcal{T} \tag{4}$$

The following lemma shows that these inequalities define a valid upper bound for variables u_t^i, coupling all states j together.

Proposition 1. *Strengthened Upper Bound inequalities* (4) *are valid for* $\mathcal{P}^{MT}$.

Proof. If $x_t^i - x_{t-1}^i = 1$ then $u_t^i = 1$ by (2a) therefore (SUB) is valid. Similarly, if $x_t^i - x_{t-1}^i = 0$ then $u_t^i = 0$ by (2b) and (2c) therefore (SUB) is valid.

If $x_t^i - x_{t-1}^i = -1$, it means that state i has been deactivated at time t: by (16), another state j should have been switched on at time t, thus $\sum_{j \neq i} u_t^j = 1$ which proves the validity of (SUB). □

1.2 Complete polyhedral description of the minimum-time polytope

In order to prove that strengthened upper bound inequalities, along with turn-on/turn-off inequalities, give a complete linear description of $\mathcal{P}^{MT}$, we introduce a few definitions.

Definition 1. *Let polytope $\mathcal{Q}_{n,T}$ be defined by inequalities* (1) (2a) (2b) (2c)*, turn-on* (3) *and strengthened upper bound inequalities* (4)*.*

$$\mathcal{Q}_{n,T} = \left\{ (x,u) \in [0,1]^{2n(T-1)} \mid (1),\ (2a),\ (2b),\ (2c),\ (3),\ (4) \right\}$$

Let $\{a_s,\ s \in S\}$ be a set of integral points of $\mathcal{Q}_{n,T}$. We introduce the following notations.

We denote by $x_t^i(a_s)$ (resp. $u_t^i(a_s)$) the coordinate x_t^i (resp. u_t^i) of point a_s.

We introduce set $S_t^i \subseteq S$ containing points of S such that state i is on from time $t - L^i + 1$ to time t, for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$.

$$S_t^i = \left\{ s \in S \mid x_t^i(a_s) = 1,\ \sum_{t'=t-L^i+1}^{t} u_t^i(a_s) = 0 \right\}$$

Subset S_t^i contains the points a_s such that state i is ready to be deactivated at $t+1$ (*i.e*, minimum-time constraints would indeed be satisfied if i were deactivated at $t+1$).

We also introduce set $\overline{S}_t^i \subseteq S$ containing points of S such that state i is off at time t, for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$, and no other state j is switched on on interval $[t - L^j + 1, t]$:

$$\overline{S}_t^i = \left\{ s \in S \mid x_t^i(a_s) = 0,\ \sum_{t'=t-L^j+1}^{t} u_t^j(a_s) = 0,\ \forall j \in \mathcal{N} \right\}$$

Subset $\overline{S}_t^i$ contains the points a_s such that state i is ready to be activated at $t+1$ ($i.e$, min-up time constraints would be satisfied for other states j if i were activated at $t+1$).

We introduce set $S_t^{i,u} \subseteq S$ containing points of S such that state i is switched on at time t, for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$.

$$S_t^{i,u} = \left\{ s \in S \mid u_t^i(a_s) = 1 \right\}$$

Finally, for a given point (x, u) we use the notation w to denote the following values:

$$w_t^i = x_{t-1}^i - x_t^i + u_t^i \tag{5}$$

Note that $w_t^i = 1$ would correspond to switching off state i at time t.

Now we prove three useful lemmas to show that $\mathcal{Q}_{n,T}$ is integral.

Lemma 1. *For any point (x, u) in $\mathcal{Q}_{n,T}$, the following equality holds:*

$$\sum_{i=0}^{n} w_t^i = \sum_{i=0}^{n} u_t^i \tag{6}$$

Proof. We have $\sum_{i=0}^{n} x_{t-1}^i - x_t^i + u_t^i = 1 - 1 + \sum_{i=0}^{n} u_t^i$ from inequality (16). □

Lemma 2. *For any point (x, u) in $\mathcal{Q}_{n,T}$, the following inequality holds:*

$$\sum_{i \neq j} w_t^i \geq u_t^j \qquad \forall j \in \mathcal{N}. \tag{7}$$

Proof. By (5), $\sum_{i \neq j} w_t^i = \sum_{i \neq j} (u_t^i + x_{t-1}^i - x_t^i)$. Then applying (16) for t and $t-1$, we obtain $\sum_{i \neq j} w_t^i = x_t^j - x_{t-1}^j + \sum_{i \neq j} u_t^i$. By (SBU) inequality (4), we obtain the result. □

For a point (x, u) in $\mathcal{Q}_{n,T}$, consider oriented graph $G_{x,u} = (V, A)$, where $V = \{s\} \cup \{p\} \cup \{W^i,\ U^i,\ \forall i \in \mathcal{N}\}$ and $A = \{(s, W^i)\ \forall i\} \cup \{(W^i, U^j),\ \forall i \neq j\} \cup \{(U^i, p)\ \forall i\}$.

Figure 3 shows the form of graph G for a 4-state system. For each $i \in \mathcal{N}$, arc (s, W^i) has capacity w_T^i and arc (U^i, t) has capacity u_T^i.

Lemma 3. *Consider a point (x, u) in $\mathcal{Q}_{n,T}$. Let $Q = \sum_{i \in \mathcal{N}} w_T^i$. There exists an s–p flow of value Q in graph $G_{x,u}$.*

Proof. Suppose the maximum s–p flow were of value $Q' < Q$. Then it would mean that at least one arc (s, W^j) is not saturated for a given j. As $Q = \sum_{i \in \mathcal{N}} u_T^i$ by Lemma 4, some arcs (U^i, t) are not saturated either. If arc (U^i, p), with $i \neq j$ is not saturated, then the flow can be increased on arcs (s, W^j) and (W^j, U^i), therefore Q' would not be a maximum flow. Thus, among arcs (U^i, t), $i \in \mathcal{N}$, only arc (U^j, p) is not saturated. Thus, for a given postitive flow f on an arc $(W^i, U^{i'})$, with $i \neq j$, $i' \neq j$, flow f can be (at least partly) re-oriented to arc (W^i, U^j). Therefore, arc $(U^{i'}, p)$ would not be saturated anymore, and we could increase the flow on arcs (s, W^j), $(W^j, U^{i'})$, and $(U^{i'}, p)$. This leads to a contradiction. □

Theorem 1. *Polytope $\mathcal{Q}_{n,T}$ is integral.*

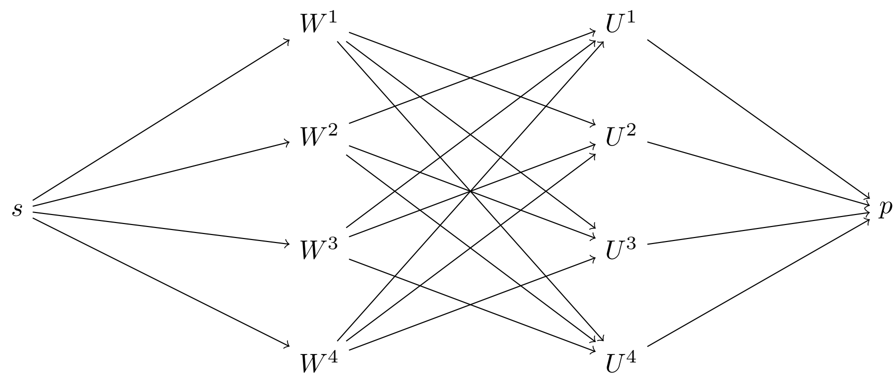


Figure 3: Example of a graph $G_{x,u}$ with $n = 4$

Proof. We prove the following result by induction on T: for any point (x, u) of $\mathcal{Q}_{n,T}$, there exists a set $\{a_s,\ s \in S\}$ of integral points $a_s \in \mathcal{Q}_{n,T}$ such that (x, u) can be written as a convex combination of points a_s, such that: $(x, u) = \sum_{s\in S} \lambda_s a_s$ where $\sum_{s\in S} \lambda_s = 1$.

For $T = 1$, the result trivially holds as variables u are non-zero only when $T \geq 2$.

Suppose the result is true for $T-1$. We first prove (i) for T. Consider a point (x, u) of $\mathcal{Q}_{n,T}$. By induction hypothesis, the restriction $(x, u)_{T-1}$ of (x, u) to the first $T-1$ time steps can be written as a convex combination of integral points $a_s \in \mathcal{Q}_{n,T-1}$, $s \in S$ such that $(x, u)_{T-1} = \sum_{s\in S} \lambda_s a_s$ where $\sum_{s\in S} \lambda_s = 1$. We will show that each point a^s can be extended to time step T so that for each a_s we obtain several points $b^p_s = (a_s, (x^p_{s,T}, u^p_{s,T})) \in \mathcal{Q}_{n,T}$ satisfying: $(x, u) = \sum_{s\in S} \sum_{p\in P_s} \lambda^p_s b^p_s$ where $\sum_{p\in P_s} \lambda^p_s = \lambda_s$.

For each state i, a fraction u^i_T of state i is activated at time T. In this case, a fraction u^i_T of integral points a_s, $s \in \cup_{j\neq i} S^j_{T-1}$ must be extended so that state i is activated at time T (*i.e.*, extended so that the resulting points satisfy $x^i_{T-1} = 0$ and $x^i_T = 1$). The quantity of eligible of points in S is

$$Q^i_u = \sum_{s\in\cup_{j\neq i} S^j_{T-1}} \lambda_s = \sum_{s\in \overline{S}^i_t} \lambda_s = 1 - x^i_{T-1} - \sum_{j\neq i} \sum_{t=T-L^j+1}^{T-1} \sum_{s\in S^u_{j,t}} \lambda_s = 1 - x^i_{T-1} - \sum_{j\neq i} \sum_{t=T-L^j+1}^{T-1} u^j_t$$

by induction hypothesis. By inequality (3), and then (16):

$$Q^i_u \geq 1 - x^i_{T-1} + \sum_{j\neq i} (u^j_T - x^j_T) \geq 1 - \sum_{j\neq i} x^j_T - x^i_{T-1} + \sum_{j\neq i} u^j_T = x^i_T - x^i_{T-1} + \sum_{j\neq i} u^j_T$$

Finally, by (4), we obtain: $Q^i_u \geq u^i_T$.

Moreover, a fraction w_T^i of state i is deactivated at time T. In this case, a fraction of integral points a_s, $s \in S_{T-1}^i$ must be extended so that state i is switched off at time T (*i.e.*, extended so that the resulting points satisfy $x_T^i = 0$). The quantity of eligible points is

$$Q_w^i = \sum_{s \in S_{T-1}^i} \lambda_s = x_{T-1}^i - \sum_{t=T-L^i+1}^{T-1} u_t^i \geq x_{T-1}^i - x_T^i + u_T^i$$

by induction hypothesis and inequality (3). Finally, using (5) we obtain $Q_w^i \geq w_T^i$.

We have shown that for each state i, there is a sufficient fraction $Q_u^i \geq u_T^i$ (resp. $Q_w^i \geq w_T^i$) of points ready to activate (resp. deactivate) state i. It remains to decide how each point a_s will be extended. To do this, we argue that an extended solution can be obtained from a flow in graph $G_{x,u}$

□

2 The No-Spike polytope

In this section, we consider a system subject to no-spike constraints instead of minimum-time constraints. To formulate these constraints, exclusion constraints (1) are replaced with the following order constraints:

$$x_t^0 = 1 \qquad \text{and} \qquad x_t^i \leq x_t^{i-1} \qquad \forall t \in \mathcal{T}, \qquad \forall i \in \{1, ..., n\} \tag{8}$$

In this case, variables x take on a new meaning: $x_t^i = 1$ if there is a state $j \geq i$ such that all states from 1 to j are on at time t. The states of the system are then modeled as incremental, in the sense that if state i is on at time t, then all states $k < i$ must also be on at time t.

Note that when featuring incremental variables x, the extended turn-on inequalities (3) take another meaning. Specifically, they enforce that when state i is switched on, the system must remain on states $j \geq i$ during L^i time steps. Therefore, in this incremental state context, the min-up constraints (3) exactly enforce no-up-spike constraints.

We can similarly formulate no-down-spike constraints as *extended turn-off* inequalities:

$$\sum_{t'=t-\ell^i+1}^{t} u_{t'}^i \leq 1 - x_{t-\ell^i}^i \tag{9}$$

These inequalities enforce that when state i is switched off, then the system must remain on states $j < i$ during ℓ^i time steps.

We study the resulting *no-spike polytope* $\mathcal{P}^{NS}$ with symmetric min-up/min-down times $L^i = \ell^i = L$, for each $i \in \mathcal{N}$.

$$\mathcal{P}^{NS} = conv\left\{(x,u) \in \{0,1\}^{(n\times T)\times(n\times T-1)} \mid \text{(2a)},\ \text{(2b)},\ \text{(2c)},\ \text{(3)},\ \text{(8)},\ \text{(9)}\right\}$$

In the following sections, we introduce several families of valid inequalities for $\mathcal{P}^{NS}$. We then prove that these inequalities, along with formulation inequalities, yield a complete description of the minimum-time polytope.

2.1 Valid inequalities

In this section, we introduce two linear-size families of valid inequalities for $\mathcal{P}^{NS}$, one that generalizes (8) using u variables, and another that captures the coupling between (3) (resp. (9)) and (8).

Definition 2 (Generalized order inequalities)**.** *For* $i \in \{1, ...n-1\}$, $t \in \{1, ..., T-1\}$ *:*

$$x_t^{i+1} + u_{t+1}^{i+1} \leq x_t^i + u_{t+1}^i \tag{10}$$

$$x_t^{i+1} + u_t^i \leq x_t^i + u_t^{i+1} \tag{11}$$

Proposition 2. *Generalized order inequalities* (10) (11) *are valid for* $\mathcal{P}^{NS}$*.*

Proof. Note that by constraint (8), $x_t^i - x_t^{i+1} \geq 0$. If $x_t^i - x_t^{i+1} = 1$, then both inequality are valid. Otherwise, if $x_t^i - x_t^{i+1} = 0$, it means that states i and $i+1$ are either both activated or both deactivated at time t.

We first prove the validity of (10) in the case where $x_t^i - x_t^{i+1} = 0$. If $u_{t+1}^{i+1} = 1$, meaning that state $i+1$ is switched on at time $t+1$, then state i is also switched on at time $t+1$, so $u_{t+1}^i = 1$.

Now we prove the validity of (11) in the case where $x_t^i - x_t^{i+1} = 0$. If $u_t^i = 1$, meaning that state i is switched on at time t, then state $i+1$ is also switched on at time t, as $x_t^i - x_t^{i+1} = 0$, thus $u_t^{i+1} = 1$. □

Now we introduce inequalities generalizing no-up-spike (3) and no-down-spike (9) constraints.

Note that by convention, $x_t^{n+1} = 0$, $u_t^{n+1} = 0$ and $x_t^0 = 1$, $u_t^0 = 0$, for all $t \in \mathcal{T}$.

Definition 3 (Generalized turn-on inequalities)**.** *For* $i \in \{1, ...n\}$, $t \in \{L+1, ..., T\}$ *and* $k \in \{1, ..., L-1\}$*:*

$$\sum_{t'=t-k}^{t} u_{t'}^i \leq x_t^i + \sum_{t'=t-k}^{t} u_{t'}^{i+1} - x_t^{i+1} \tag{12}$$

Proposition 3. *Generalized turn-on inequalities* (12) *are valid for* $\mathcal{P}^{NS}$*.*

Proof. Note that by constraint (8), $x_t^i - x_t^{i+1} \geq 0$. If $x_t^i - x_t^{i+1} = 1$, then the inequality is valid: indeed, $\sum_{t'=t-k}^{t} u_{t'}^i \leq 1$ otherwise the minimum-time constraint would not be satisfied. We now consider the case where $x_t^i - x_t^{i+1} = 0$, meaning that states i and $i+1$ are either both activated or both deactivated at time t. If $\sum_{t'=t-k}^{t} u_{t'}^i = 0$ then the inequality is trivially valid. Otherwhise, if $\sum_{t'=t-k}^{t} u_{t'}^i = 1$, it means that state i has been switched on at some time $t' \in [t-k, t]$. Since $k \leq L-1$, by the minimum-time constraint, state i is still activated at time t. As $x_t^i - x_t^{i+1} = 0$, state $i+1$ is therefore also activated at time t. As state i was not activated at time $t'-1$, state $i+1$ was not activated neither at time $t'-1$. Therefore, there exists a time $t'' \in [t', t]$ at which state $i+1$ has been activated. Thus, $\sum_{t'=t-k}^{t} u_{t'}^{i+1} = 1$.

□

Proposition 4. *Generalized turn-on inequalities dominate turn-on inequalities.*

Proof. For $k = L-1$, the generalized turn-on inequality has the form:

$$\sum_{t'=t-L+1}^{t} u_{t'}^{i} \leq x_t^i + \sum_{t'=t-L+1}^{t} u_{t'}^{i+1} - x_t^{i+1}$$

It corresponds to the turn-on inequality for state i, with the addition of $\sum_{t'=t-L+1}^{t} u_{t'}^{i+1} - x_t^{i+1}$ to the right hand side. Note that $\sum_{t'=t-L+1}^{t} u_{t'}^{i+1} - x_t^{i+1} \leq 0$ by the turn-on inequality for state $i+1$. Therefore, the generalized turn-on inequality dominates the turn-on inequality, for each state $i \in \mathcal{N}$. □

Definition 4 (Generalized turn-off inequalities). *For $i \in \{1, ...n\}$, $t \in \{L+1, ..., T\}$ and $k \in \{1, ..., L-1\}$:*

$$x_{t-L}^{i} + \sum_{t'=t-k}^{t} u_{t'}^{i} \leq x_{t-L}^{i-1} + \sum_{t'=t-k}^{t} u_{t'}^{i-1} \tag{13}$$

Proposition 5. *Generalized turn-off inequalities* (13) *are valid for $\mathcal{P}^{NS}$.*

Proof. The proof follows the same pattern as the proof of Proposition 3. □

Proposition 6. *Generalized turn-off inequalities dominate turn-off inequalities.*

Proof. The proof follows the same pattern as the proof of Proposition 4. □

Definition 5. *Let polytope $\mathcal{Q}_{n,T}^{NS}$ be defined by generalized turn-on inequalities* (12)*, generalized turn-off inequalities* (13)*, generalized order inequalities* (10) (11) *along with order inequalities* (8) *and inequalities* (2a) (2b) (2c)*:*

$$\mathcal{Q}_{n,T}^{NS} = \left\{ (x,u) \in [0,1]^{2n(T-1)} \mid \text{(2a)(2b)(2c)(8)(10)(11)(12)(13)} \right\}$$

2.2 Complete polyhedral description of the no-spike polytope

In this section, we prove that polytope $\mathcal{Q}_{n,T}^{NS}$ is integral, therefore inequalities (2a) (2b) (2c) (8) (10) (11) (12) (13) give a complete linear description of the no-spike polytope $\mathcal{P}^{NS}$.

In order to prove the integrality of $\mathcal{Q}_{n,T}^{NS}$, we first apply a bijective transformation: we obtain a new polytope $\widetilde{\mathcal{Q}}_{n,T}^{NS}$, which we prove to be integral.

2.2.1 Linear bijective transformation of $\mathcal{Q}_{n,T}^{NS}$

We consider linear bijection ϕ transforming variables (x,u) to variables $(\widetilde{x}, \widetilde{u}, \widetilde{w})$ as follows :

$$\phi : (x,u) \to (\widetilde{x}, \widetilde{u}, \widetilde{w})$$

where for each $i \in \{0, ..., n\}$ and $t \in \mathcal{T}$,

$$\begin{aligned} \widetilde{x}_t^i &= x_t^i - x_t^{i+1} \\ \widetilde{u}_t^i &= u_t^i - u_t^{i+1} \\ \widetilde{w}_t^i &= w_t^i - w_t^{i+1} \end{aligned} \tag{14}$$

and

$$w_t^i = u_t^i + x_t^{i-1} - x_t^i$$

Note that $\widetilde{x}_t^i \geq 0$ by order constraints, but $\widetilde{u}_t^i \in [-1,1]$.

Inverse bijection ϕ^{-1} can be defined as follows:

$$\phi^{-1} : (\widetilde{x}, \widetilde{u}, \widetilde{w}) \to (x, u)$$

where

$$\begin{array}{c} \sum_{j=i}^{n} \widetilde{x}_t^i = x_t^i \\ \sum_{j=i}^{n} \widetilde{u}_t^i = u_t^i \\ \sum_{j=i}^{n} \widetilde{w}_t^i = w_t^i \end{array} \tag{15}$$

We can easily prove the following lemmas.

Lemma 4. *For* $(x, u) \in \mathcal{Q}_{n,T}^{NS}$ *and* $(\widetilde{x}, \widetilde{u}, \widetilde{w})$ *satisfying* (14),

$$\widetilde{w}_t^i = \widetilde{u}_t^i + \widetilde{x}_t^{i-1} - x_t^i$$

Lemma 5. *With bijection* ϕ*, generalized order inequalities* (10) *and* (11) *rewrite*

$$-\widetilde{x}_{t-1}^i \leq \widetilde{u}_t^i \leq \widetilde{x}_t^i$$

Lemma 6. *With bijection* ϕ*, for* $i \in \{1, ...n\}$*,* $t \in \{L+1, ..., T\}$ *and* $k \in \{1, ..., L-1\}$*, generalized turn-on inequalities rewrite*

$$\sum_{t'=t-k}^{t} \widetilde{u}_{t'}^i \leq \widetilde{x}_t^i$$

Lemma 7. *With bijection* ϕ*, for* $i \in \{1, ...n\}$*,* $t \in \{L+1, ..., T\}$ *and* $k \in \{1, ..., L-1\}$*, generalized turn-off inequalities rewrite*

$$\widetilde{x}_{t-k-1}^i + \sum_{t'=t-k}^{t} \widetilde{u}_{t'}^i \geq 0$$

Bijection ϕ transforms the incremental formulation with variables (x, u) into a multiple choice formulation with variables $(\widetilde{x}, \widetilde{u}, \widetilde{w})$. Indeed, we prove that along bijection ϕ, polytope $\mathcal{Q}_{n,T}^{ord}$ is transformed to $\widetilde{\mathcal{Q}}^{NS}_{n,T}$:

$$\widetilde{\mathcal{Q}}^{NS}_{n,T} = \left\{ \begin{array}{lr} \displaystyle\sum_{i=0}^{n} \widetilde{x}^i_t = 1 & (16) \\ \displaystyle\sum_{i=0}^{n} \widetilde{u}^i_t = \sum_{i=0}^{n} \widetilde{w}^i_t = 0 & (17) \\ \displaystyle\sum_{j=i}^{n} \widetilde{u}^j_t \geq 0 \qquad \text{and} \qquad \sum_{j=i}^{n} \widetilde{w}^j_t \geq 0 & (18) \\ \widetilde{w}^i_t = \widetilde{u}^i_t + \widetilde{x}^i_{t-1} - \widetilde{x}^i_t & (19) \\ -\widetilde{x}^i_{t-1} \leq \widetilde{u}^i_t \leq \widetilde{x}^i_t & (20) \\ \displaystyle\sum_{t'=t-k}^{t} \widetilde{u}^i_{t'} \leq \widetilde{x}^i_t \qquad \forall k \in \{1, ..., L-1\} & (21) \\ \displaystyle\widetilde{x}^i_{t-k-1} + \sum_{t'=t-k}^{t} \widetilde{u}^i_{t'} \geq 0 \qquad \forall k \in \{1, ..., L-1\} & (22) \\ \widetilde{x}^i_t \in [0,1], \ \widetilde{u}^i_t \in [-1,1] & (23) \end{array} \right\}$$

Theorem 2.

$$\phi(\mathcal{Q}^{ord}_{n,T}) = \widetilde{\mathcal{Q}}^{NS}_{n,T}$$

Proof. We first prove the inclusion $\phi(\mathcal{Q}^{ord}_{n,T}) \subseteq \widetilde{\mathcal{Q}}^{NS}_{n,T}$. Let $(x,u) \in \mathcal{Q}^{ord}_{n,T}$. Let $(\widetilde{x}, \widetilde{u}, \widetilde{w}) = \phi((x,u))$. We show that $(\widetilde{x}, \widetilde{u}, \widetilde{w}) \in \widetilde{\mathcal{Q}}^{NS}_{n,T}$. By Lemmas 4 to 7, $(\widetilde{x}, \widetilde{u}, \widetilde{w})$ satisfy (in)equalities (19), (20), (21) and (22). Inequalities (16) and (17) are obtained for each t by summing equalities (14) from $i = 0$ to n, given that $x^i_0 = 1$ and $u^i_0 = w^i_0 = 0$. Inequalities (18) are obtained for each t and eahc i by summing equalities (14) from $j = i$ to n, given that $u^i_t \geq 0$ and $w^i_t \geq 0$.

Finally we prove $\widetilde{\mathcal{Q}}^{NS}_{n,T} \subseteq \phi(\mathcal{Q}^{ord}_{n,T})$. Let $(\widetilde{x}, \widetilde{u}, \widetilde{w}) \in \widetilde{\mathcal{Q}}^{NS}_{n,T}$ and let $(x,u) = \phi^{-1}((\widetilde{x}, \widetilde{u}, \widetilde{w}))$. We show that $(x,u) \in \mathcal{Q}^{ord}_{n,T}$. From $\sum_{j=i}^{n} \widetilde{w}^j_t \geq 0$ we can deduce $\sum_{j=i}^{n} \widetilde{u}^j_t + \widetilde{x}^j_{t-1} - \widetilde{x}^j_t \geq 0$ therefore inequality (2a) is satisfied by (x,u). From $\widetilde{u}^i_t \leq \widetilde{x}^i_t$ we obtain $\sum_{j=i}^{n} \widetilde{u}^j_t \leq \sum_{j=i}^{n} \widetilde{x}^j_t$ therefore inequality (2b) is satisfied by (x,u). From $-\widetilde{x}^i_{t-1} \leq \widetilde{u}^i_t$ we obtain $\sum_{j=0}^{i-1} \widetilde{x}^j_{t-1} + \widetilde{u}^j_t \geq 0$. Using (16) and (17), it transforms into $\sum_{j=i}^{n} \widetilde{x}^j_{t-1} + \widetilde{u}^j_t \leq 1$, showing that inequality (2c) is satisfied by (x,u).

□

Interpretation of variables $(\widetilde{x}, \widetilde{u}, \widetilde{w})$ Bijection ϕ preserves integrality, therefore any integer solution $(\widetilde{x}, \widetilde{u}, \widetilde{w})$ corresponds to an integer solution (x,u), and conversely.

Decision variables $\widetilde{x} \in \{0,1\}^{nT}$ can be interpreted as follows:

$$\widetilde{x}^i_t = \begin{cases} 1 & \text{if state } i \text{ is used at time } t \\ 0 & \text{otherwise} \end{cases}$$

where at each time t, **exactly one** state is used by the system (by equality (16)).

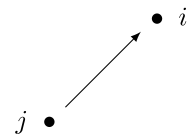
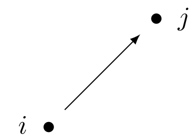
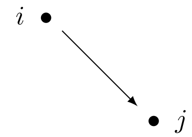
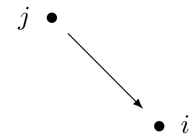
	= 1	= -1
$\widetilde{u}_t^i$	 i is upwardly activated ($j < i$)	 i is upwardly deactivated ($j > i$)
$\widetilde{w}_t^i$	 i is downwardly deactivated ($j < i$)	 i is downwardly activated ($j > i$)

Table 1: Meaning of variables $\widetilde{u}$ and $\widetilde{w}$ depending on their values

Variables $\widetilde{w} \in \{-1, 0, 1\}^{n(T-1)}$ take the following values:

$$\widetilde{w}_t^i = \begin{cases} 1 & \text{if } w_t^i = 1 \text{ and } w_t^{i+1} = 0 \\ -1 & \text{if } w_t^i = 0 \text{ and } w_t^{i+1} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if $\widetilde{w}_t^i = 1$, it means that state i is used at time $t-1$ but is *downwardly deactivated* at time t, meaning that from $t-1$ to t, there is a transition from i to some state $j < i$. If $\widetilde{w}_t^i = -1$, it means that state i is not used at time $t-1$ but is *downwardly activated* at time t, meaning that from $t-1$ to t, there is a transition from some state $j > i$ to state i.

Variables $\widetilde{u} \in \{-1, 0, 1\}^{n(T-1)}$ take the following values:

$$\widetilde{u}_t^i = \begin{cases} 1 & \text{if } u_t^i = 1 \text{ and } u_t^{i+1} = 0 \\ -1 & \text{if } u_t^i = 0 \text{ and } u_t^{i+1} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if $\widetilde{u}_t^i = 1$, it means that state i is not used at time $t-1$ but is *upwardly activated* at time t, meaning that from $t-1$ to t, there is a transition from some state $j < i$ to state i. If $\widetilde{u}_t^i = -1$, it means that state i is used at time $t-1$ but is *upwardly deactivated* at time t, meaning that from $t-1$ to t, there is a transition from i to some state $j > i$.

Table 1 synthesizes the meanings of variables $\widetilde{u}$ and $\widetilde{w}$ depending on their values.

2.2.2 Integrality of $\widetilde{\mathcal{Q}}_{n,T}^{NS}$

In this section, we show that $\widetilde{\mathcal{Q}}_{n,T}^{NS}$ is an integral polytope.

Let $\mathcal{S} = \{a_s,\ s \in S\}$ be a set of integral points of $\widetilde{\mathcal{Q}}_{n,T}^{NS}$. We introduce the following notations.

We denote by $\widetilde{x}_t^i(a_s)$ (resp. $\widetilde{u}_t^i(a_s)$, $\widetilde{w}_t^i(a_s)$) the coordinate $\widetilde{x}_t^i$ (resp. $\widetilde{u}_t^i$, $\widetilde{w}_t^i$) of point $a_s \in \mathcal{S}$.

We introduce set $S_t^i \subseteq \mathcal{S}$ containing points of $\mathcal{S}$ such that state i is on at time t.

$$S_t^i = \{s \in S \mid \widetilde{x}_t^i(a_s) = 1\}$$

Subset S_t^i contains the points a_s such that state i is used at time t.

The following subsets of $\mathcal{S}$ are also defined:

$$S_{[t',t]}^{i,\nearrow a} = \{s \in S \mid \exists k \in [t',t] \text{ s. t. } \widetilde{u}_k^i(a_s) = 1 \text{ and } \widetilde{x}_t^i(a_s) = 1\}$$

$$S_{[t',t]}^{i,\searrow a} = \{s \in S \mid \exists k \in [t',t] \text{ s. t. } \widetilde{w}_k^i(a_s) = -1 \text{ and } \widetilde{x}_t^i(a_s) = 1\}$$

Subset $S_{[t',t]}^{i,\nearrow a}$ (resp. $S_{[t',t]}^{i,\searrow a}$) contains the points such that state i is upwardly activated (resp. downwardly activated) at some time $k \in [t',t]$, and such that state i is still used at time t.

We now introduce useful lemmas to prove integrality of $\widetilde{\mathcal{Q}}_{n,T}^{NS}$.

Lemma 8. *For any point $(\widetilde{x}, \widetilde{u}, \widetilde{w})$ of $\widetilde{\mathcal{Q}}_{n,T}^{NS}$, the following inequality holds:*

$$\forall t \in \mathcal{T},\ \forall i \in \mathcal{N}, \qquad \widetilde{x}_{t-1}^i - \max_{k\in\{1,\ldots,L-1\}} \left(0, \sum_{t'=t-k}^{t-1} \widetilde{u}_{t'}^i\right) \geq \widetilde{w}_t^i$$

Proof. We consider two cases.

- Case 1: $\max_{k\in\{1,\ldots,L-1\}} \left(0, \sum_{t'=t-k}^{t-1} \widetilde{u}_{t'}^i\right) = 0$.

  From (20), $\widetilde{u}_t^i \leq \widetilde{x}_t^i$ holds. Replacing $\widetilde{u}_t^i$ by the expression given by (19), we obtain the result.

- Case 2: $\max_{k\in\{1,\ldots,L-1\}} \left(0, \sum_{t'=t-k}^{t-1} \widetilde{u}_{t'}^i\right) = \sum_{t'=t-\overline{k}}^{t-1} \widetilde{u}_{t'}^i$ for some $\overline{k} \in \{1, ..., L-1\}$.

  From (21), $\widetilde{x}_t^i \geq \widetilde{u}_t^i + \sum_{t'=t-\overline{k}}^{t-1} \widetilde{u}_{t'}^i$ holds. Replacing $\widetilde{u'}_t^i$ by the expression given by (19), we obtain the result.

□

Lemma 9. *For any point $(\widetilde{x}, \widetilde{u}, \widetilde{w})$ of $\widetilde{\mathcal{Q}}_{n,T}^{NS}$, the following inequality holds:*

$$\forall t \in \mathcal{T},\ \forall i \in \mathcal{N}, \qquad \widetilde{x}_{T-1}^i - \max_{k\in\{1,\ldots,L-1\}} \left(0, \sum_{t'=T-k}^{T-1} \widetilde{w}_{t'}^i\right) \geq -\widetilde{u}_t^i$$

Proof. We consider two cases.

- Case 1: $\max_{k\in\{1,\ldots,L-1\}} \left(0, \sum_{t'=t-k}^{t-1} \widetilde{w}_{t'}^i\right) = 0$.

  In this cas the result follows directly from (20).

- Case 2: $\max_{k\in\{1,\ldots,L-1\}} \left(0, \sum_{t'=t-k}^{t-1} \widetilde{w}_{t'}^i\right) = \sum_{t'=t-\overline{k}}^{t-1} \widetilde{w}_{t'}^i$ for some $\overline{k} \in \{1, ..., L-1\}$.

  From (21), $\widetilde{x}_{t-k-1}^i + \widetilde{u}_t^i + \sum_{t'=t-\overline{k}}^{t-1} \widetilde{u}_{t'}^i \geq 0$ holds. Replacing $\widetilde{u}_{t'}^i$ by the expression given by (19), we obtain the result.

□

Theorem 3. *Polytope $\widetilde{\mathcal{Q}}^{NS}_{n,T}$ is integral.*

Proof. We prove the following result by induction on T: for any point $(\widetilde{x},\widetilde{u},\widetilde{w})$ of $\widetilde{\mathcal{Q}}^{NS}_{n,T}$, $(\widetilde{x},\widetilde{u},\widetilde{w})$ can be written as a convex combination of integral points $a_s \in \mathcal{S}$, such that :

(i) $(\widetilde{x},\widetilde{u},\widetilde{w}) = \sum_{s\in S} \lambda_s a_s$ where $\sum_{s\in S}\lambda_s = 1$

(ii) $\forall t \in \{1,...,T\},\ \forall \ell \in \{1,...,L\},$ $\sum_{s\in S^{i,\nearrow a}_{[t-\ell+1,t]}} \lambda_s = \max_{k\in\{1,...,\ell\}}\left(0, \sum_{t'=t-k+1}^{t} \widetilde{u}^i_{t'}\right)$

(iii) $\forall t \in \{1,...,T\},\ \forall \ell \in \{1,...,L\},$ $\sum_{s\in S^{i,\searrow a}_{[t-\ell+1,t]}} \lambda_s = \max_{k\in\{1,...,\ell\}}\left(0, \sum_{t'=t-k+1}^{t} \widetilde{w}^i_{t'}\right)$

For $T=1$, the results trivially hold as variables $\widetilde{u}$ and $\widetilde{w}$ are non-zero only when $T \geq 2$.

Suppose the result is true for $T-1$.

We first prove *(i)* for T. Consider a point $(\widetilde{x},\widetilde{u},\widetilde{w})$ of $\widetilde{\mathcal{Q}}^{NS}_{n,T}$. By induction hypothesis, the restriction $(\widetilde{x},\widetilde{u},\widetilde{w})_{T-1}$ of $(\widetilde{x},\widetilde{u},\widetilde{w})$ to the first $T-1$ time steps can be written as a convex combination of integral points $a_s \in \widetilde{\mathcal{Q}}^{ord}_{n,T-1}$, $s \in S$: $(\widetilde{x},\widetilde{u},\widetilde{w})_{T-1} = \sum_{s\in S}\lambda_s a_s$ where $\sum_{s\in S}\lambda_s = 1$. We will show that each point a^s can be extended to time step T so that for each a_s we obtain several points $\overline{a}^p_s = (a_s,(x^p_{s,T},u^p_{s,T},w^p_{s,T})) \in \widetilde{\mathcal{Q}}^{NS}_{n,T}$ satisfying: $(\widetilde{x},\widetilde{u},\widetilde{w}) = \sum_{s\in S}\sum_{p\in P_s}\lambda^p_s b^p_s$ where $\sum_{p\in P_s}\lambda^p_s = \lambda_s$.

For each state i, (19) gives $\widetilde{x}^i_t = \widetilde{x}^i_{t-1} + \widetilde{u}^i_t - \widetilde{w}^i_t$. Therefore, from time $T-1$ to time T :

- If $\widetilde{w}^i_T > 0$: a fraction $\widetilde{w}^i_T$ of state i is downwardly deactivated at time T in point $(\widetilde{x},\widetilde{u},\widetilde{w})$.

  In this case, a fraction $\widetilde{w}^i_T$ of points a_s, $s \in S^i_{T-1}$ must be extended so that state i is downwardly deactivated at time T (*i.e.*, extended so that the resulting points satisfy $x^j_T = 0$ for each $j \geq i$). Not all points a_s, $s \in S^i_{T-1}$, can be extended in such a way: only those in subset $S^i_{T-1}\backslash S^{i,\nearrow a}_{[T-L+1,T-1]}$ are eligible. Indeed, if state i has been upwardly activated at some time $t' \geq T-L+1$ in point a_s, then state i can not yet be downwardly deactivated at time T, by no-up-spike constraints.

  The eligible fraction of points is therefore:

  $$Q^{i,\searrow d}_{T-1} = \widetilde{x}^i_{T-1} - \sum_{s\in S^{i,\nearrow a}_{[T-\ell+1,T-1]}} \lambda_s$$

  As *(ii)* holds for $T-1$ we have in particular that

  $$\sum_{s\in S^{i,\nearrow a}_{[t-\ell+1,t-1]}} \lambda_s = \max_{k\in\{1,...,L-1\}}\left(0, \sum_{t'=T-k}^{T-1} \widetilde{u}^i_{t'}\right)$$

  Then $Q^{i,\searrow d}_{T-1} = \widetilde{x}^i_{T-1} - \max_{k\in\{1,...,L-1\}}\left(0, \sum_{t'=T-k}^{T-1} \widetilde{u}^i_{t'}\right)$

  By Lemma 8, $Q^{i,\searrow d}_{T-1} \geq \widetilde{w}^i_T$, thus there is a sufficient fraction of eligible points for downward deactivation of state i at time T. The downward deactivation of state i is performed in priority, among eligible points, for points such that state i had been downwardly activated the most recently.

- If $\widetilde{w}_T^i < 0$: a fraction $-\widetilde{w}_T^i$ of state i is downwardly actived at time T in point $(\widetilde{x}, \widetilde{u}, \widetilde{w})$.

  In this case, a fraction $-\widetilde{w}_T^i$ of points a_s, $s \in \cup_{j>i} S_{T-1}^j$ must be extended so that state i is downwardly activated at time T (*i.e.*, extended so that the resulting points satisfy $x_T^i = 1$ and $x_T^j = 0$). Not all points a_s, $s \in \cup_{j>i} S_{T-1}^j$, can be extended in such a way; only those in subset:
  $$\bigcup_{j>i} S_{T-1}^j \backslash S_{[T-L+1,T-1]}^{j,\nearrow a}$$
  are eligible. Indeed, if state $j \geq i$ has been upwardly activated at some time $t' \geq T-L+1$ in point a_s, then state i can not yet be downwardly activated at time T. The eligible fraction of points is therefore
  $$Q_{T-1}^{i,\searrow a} = \sum_{j>i} Q_{T-1}^{j,\searrow d}$$
  We saw that $Q_{T-1}^{j,\searrow d} \geq \widetilde{w}_T^j$, therefore $Q_{T-1}^{i,\searrow a} \geq \sum_{j>i} \widetilde{w}_T^j$. From equality (15), we obtain $Q_{T-1}^{i,\searrow a} \geq -\widetilde{w}_T^i$. Thus there is a sufficient fraction of eligible points for downward activation. The downward activation of state i is performed in priority among eligible points in S_{T-1}^j, for points such that some state $j > i$ has been downwardly activated the most recently.

- If $\widetilde{u}_T^i < 0$: a fraction $-\widetilde{u}_T^i$ of state i is upwardly deactivated at time T in point $(\widetilde{x}, \widetilde{u}, \widetilde{w})$.

  In this case, a fraction $-\widetilde{u}_T^i$ of points a_s, $s \in S_{T-1}^i$ must be extended so that state i is upwardly deactivated at time T (*i.e.*, extended so that the resulting points satisfy $x_T^j = 0$ for each $j \leq i$).

  Not all points a_s, $s \in S_{T-1}^i$, can be extended in such a way: only those in subset $S_{T-1}^i \backslash S_{[T-L+1,T-1]}^{i,\searrow a}$ are eligible. Indeed, if state i has been downwardly activated at some time $t' \geq T-L+1$ in point a_s, then state i can not yet be upwardly deactivated at time T, by no-down-spike constraints. The eligible fraction of points is therefore:
  $$Q_{T-1}^{i,\nearrow d} = \widetilde{x}_{T-1}^i - \sum_{s \in S_{[T-\ell+1,T-1]}^{i,\searrow a}} \lambda_s$$
  As (iii) holds for $T-1$ we have in particular that
  $$\sum_{s \in S_{[t-\ell+1,t-1]}^{i,\searrow a}} \lambda_s = \max_{k \in \{1,\ldots,L-1\}} \left(0, \sum_{t'=T-k}^{T-1} \widetilde{w}_{t'}^i\right)$$
  Then $Q_{T-1}^{i,\nearrow d} = \widetilde{x}_{T-1}^i - \max_{k \in \{1,\ldots,L-1\}} \left(0, \sum_{t'=T-k}^{T-1} \widetilde{w}_{t'}^i\right)$

  By Lemma 9, $Q_{T-1}^{i,\nearrow d} \geq -\widetilde{u}_T^i$, thus there is a sufficient fraction of eligible points for upward deactivation of state i. The upward deactivation is performed in priority, among eligible points, for points such that state i had been downwardly activated the most recently.

- If $\widetilde{u}_T^i > 0$: a fraction $\widetilde{u}_T^i$ of state i is upwardly activated at time T in point $(\widetilde{x}, \widetilde{u}, \widetilde{w})$.

In this case, a fraction $\widetilde{u}_T^i$ of points a_s, $s \in \cup_{j<i} S_{T-1}^j$ must be extended so that state i is upwardly activated at time T (*i.e.*, extended so that the resulting points satisfy $x_T^i = 1$ and $x_T^j = 0$). Not all points a_s, $s \in \cup_{j<i} S_{T-1}^j$, can be extended in such a way; only those in subset:

$$\bigcup_{j<i} S_{T-1}^j \backslash S_{[T-L+1,T-1]}^{j,\searrow a}$$

are eligible. Indeed, if state $j \geq i$ has been upwardly activated at some time $t' \geq T-L+1$ in point a_s, then state i can not yet be downwardly activated at time T. The eligible fraction of points is therefore

$$Q_{T-1}^{i,\nearrow a} = \sum_{j<i} Q_{T-1}^{j,\nearrow d}$$

We saw that $Q_{T-1}^{j,\nearrow d} \geq -\widetilde{u}_T^j$, therefore $Q_{T-1}^{i,\nearrow a} \geq \sum_{j<i} -\widetilde{u}_T^j$.

Combining equalities (17) and (15), we obtain $\sum_{j=0}^{i} \widetilde{u}_T^j \leq 0$. It follows that $Q_{T-1}^{i,\nearrow a} \geq \widetilde{u}_T^i$. Thus there is a sufficient fraction of eligible points for upward activation. The upward activation of state i is performed in priority, among eligible points in S_{T-1}^j, for points such that states $j < i$ have been upwardly activated the most recently.

Apart from points extended in order to perform an activation or a deactivation of some state, other points a_s are extended so that the same operating state is maintained from $T-1$ to T. Thus, we obtain a set of extended points $\overline{a}_s$, $s \in \overline{S}$ such that $(\widetilde{x}, \widetilde{u}, \widetilde{w}) = \sum_{s \in \overline{S}} \lambda_s \overline{a}_s$, where $\sum_{s \in \overline{S}} \lambda_s = 1$

Let us now prove (*ii*). Let $Q = \sum_{s \in S_{[T-\ell+1,T]}^{i,\nearrow a}} \lambda_s$ for a given $\ell \in \{1, ..., L\}$. By construction of the points $\overline{a}_s$, it holds that $Q = \max\left(\widetilde{u}_T^i + \sum_{s \in S_{[T-\ell+1,T-1]}^{i,\nearrow a}} \lambda_s, 0\right)$. Indeed, if $\widetilde{u}_T^i \geq 0$, then there is a fraction $\widetilde{u}_T^i$ of points $\overline{a}_s$ that have upwardly activated i at time T. This fraction adds to the fraction of points a_s having upwardly activated point i at some time $t' \in [T-\ell+1, T-1]$, as i cannot be upwardly activated again at time T in these points, by no-spike constraints. Then, $Q = \widetilde{u}_T^i + \sum_{s \in S_{[T-\ell+1,T-1]}^{i,\nearrow a}} \lambda_s$. Otherwise if $\widetilde{u}_T^i < 0$, then there is a fraction $-\widetilde{u}_T^i$ of points a_s that have upwardly deactivated i at time T. By construction, these points are chosen in priority among those where state i had been upwardly activated the most recently, *i.e.*, among points in $S_{[T-\ell+1,T-1]}^{i,\nearrow a}$. Therefore, fraction $-\widetilde{u}_T^i$ is substracted to $S_{[T-\ell+1,T-1]}^{i,\nearrow a} \lambda_s$ until reaching 0, *i.e.* $Q = \max\left(\widetilde{u}_T^i + \sum_{s \in S_{[T-\ell+1,T-1]}^{i,\nearrow a}} \lambda_s, 0\right)$. In both cases, (*ii*) can be obtained using the induction hypothesis.

We finally prove (*iii*). Let $Q = \sum_{s \in S_{[T-\ell+1,T]}^{i,\searrow a}} \lambda_s$ for a given $\ell \in \{1, ..., L\}$. By construction of points $\overline{a}_s$, it holds that $Q = \max\left(\widetilde{w}_T^i + \sum_{s \in S_{[T-\ell+1,T-1]}^{i,\searrow a}} \lambda_s, 0\right)$. Indeed, if $\widetilde{w}_T^i \geq 0$, then there is a fraction $\widetilde{w}_T^i$ of points a_s that have downwardly deactivated i at time T. This fraction adds to the fraction of points having downwardly deactivated point i at some time $t' \in [T-\ell+1, T-1]$, as i cannot be downwardly deactivated again at time T in these points, by the no-spike constraints. Then, $Q = \widetilde{w}_T^i + \sum_{s \in S_{[T-\ell+1,T-1]}^{i,\nearrow a}} \lambda_s$ in this case. Otherwise if $\widetilde{w}_T^i < 0$, then there is a fraction $-\widetilde{w}_T^i$ of points a_s that have downwardly deactivated i at time T. By construction, these points are chosen in priority among those where state i had been downwardly activated the most recently, *i.e.*, among points in $S_{[T-\ell+1,T-1]}^{i,\searrow a}$. Therefore, fraction $\widetilde{u}_T^i$ is substracted to $S_{[T-\ell+1,T-1]}^{i,\searrow a} \lambda_s$ until reaching 0, *i.e.* $Q = \max\left(\widetilde{u}^i_T + \sum_{s \in S^{i,\nearrow a}_{[T-\ell+1,T-1]}} \lambda_s, 0\right)$. In both cases, (*iii*) can be obtained using the induction hypothesis.

□

2.2.3 Integrality of $\mathcal{Q}^{NS}_{n,T}$

Corollary 1. *Polytope* $\mathcal{Q}^{NS}_{n,T}$ *is integral.*

Proof. It can easily be seen that bijection ϕ preserves integrality. As it is also linear, it preserves extreme points as well. Therefore the integrality of $\widetilde{\mathcal{Q}}^{NS}_{n,T}$ implies the integrality of $\mathcal{Q}^{NS}_{n,T}$. □

Corollary 2 (Complete polyhedral description of $\mathcal{P}^{NS}$)**.** *Polytope* $\mathcal{P}^{NS}$ *is completely described by inequalities* (2a) (2b) (2c) (8) (10) (11) (12) (13)*.*

Proof. From propositions 2, 3 and 5, inequalities (10) (11) (12) (13) are valid for $\mathcal{P}^{NS}$, and dominate (3) and (9). By Corollary 1, it shows that the convex hull of (2a) (2b) (2c) (8) (10) (11) (12) (13) is integral, therefore proving the result. □

3 Conclusion and perspectives

Minimum-time and no-spike constraints have been defined as generalizations of min-up/min-down constraints from the literature. We have shown that these constraints could be both formulated with extended turn-on/off inequalities. For the minimum-time constraints, turn-on/off inequalities involve multiple choice variables and are thus coupled to exclusion constraints. For the no-spike constraints, turn-on/off inequalities involve incremental variables and are thus coupled to order constraints. Associated minimum-time polytope and no-spike polytope have been defined. Linear size families of valid inequalities, generalizing formulation inequalities, are introduced for both polytopes. For each polytope, these valid inequalities, along with formulation inequalities, are shown to provide a complete linear description.

As a perspective, the polyhedral study could be further developed for even more general cases: for example, the no-spike polytope has been defined here with symmetric min-up/min-down times. Even though in practice instances are usually symmetric, the non-symmetric case could also be explored. Furthermore, polyhedral couplings between minimum-time or no-spike constraints and other constraints could also be worth investigating to solve practical Unit Commitment Problems. For example, an interesting coupling would be with transition constraints enforcing that from each state, only a subset of other states are accessible. Finally, experimental results could be conducted to determine the cases when the valid inequalities proposed in this paper lead to a more efficient resolution.

References

[1] Usman Ali and Magnus Egerstedt. Optimal control of switched dynamical systems under dwell time constraints. In *53rd IEEE Conference on Decision and Control*, pages 4673–4678. IEEE, 2014.

[2] P. Bendotti, P. Fouilhoux, and C. Rottner. The min-up/min-down unit commitment polytope. *Journal of Combinatorial Optimization*, 36(3):1024–1058, 2018.

[3] Yutao Chen and Mircea Lazar. An efficient MPC algorithm for switched nonlinear systems with minimum dwell time constraints. *arXiv preprint arXiv:2002.09658*, 2020.

[4] Nicolas Dupin. Tighter MIP formulations for the discretised unit commitment problem with min-stop ramping constraints. *EURO Journal on Computational Optimization*, 5(1-2):149–176, 2017.

[5] Lei Fan and Yongpei Guan. An edge-based formulation for combined-cycle units. *IEEE Transactions on Power Systems*, 31(3):1809–1819, 2015.

[6] Lei Fan, Kai Pan, and Yongpei Guan. A strengthened mixed-integer linear programming formulation for combined-cycle units. *European Journal of Operational Research*, 275(3):865–881, 2019.

[7] A. Frangioni and C. Gentile. New MIP formulations for the single-unit commitment problems with ramping constraints. *IASI Research Report 15-06*, 2017.

[8] Grace Hechme-Doukopoulos, Sandrine Brignol-Charousset, Jérôme Malick, and Claude Lemaréchal. The short-term electricity production management problem at edf. *Optima Newsletter*, 84:2–6, 2010.

[9] Ali Heydari. Optimal switching with minimum dwell time constraint. *Journal of the Franklin Institute*, 354(11):4498–4518, 2017.

[10] Ben Knueven, Jim Ostrowski, and Jianhui Wang. The ramping polytope and cut generation for the unit commitment problem. *INFORMS Journal on Computing*, 30(4):739–749, 2018.

[11] J. Lee, J. Leung, and F. Margot. Min-up/min-down polytopes. *Discrete Optimization*, 1:77–85, 2004.

[12] P. Malkin. Minimum runtime and stoptime polyhedra. *Private Communication*, 2003.

[13] Germán Morales-España, Carlos M Correa-Posada, and Andres Ramos. Tight and compact mip formulation of configuration-based combined-cycle units. *IEEE Transactions on Power Systems*, 31(2):1350–1359, 2015.

[14] J. Ostrowski, M. F. Anjos, and A. Vannelli. Tight mixed integer linear programming formulations for the unit commitment problem. *IEEE Transactions on Power Systems*, 27(1):39–46, 2012.

[15] K. Pan and Y. Guan. A polyhedral study of the integrated minimum-up/-down time and ramping polytope. *Optimization Online*, 2016. http://www.optimization-online.org/DB_HTML/2015/08/5070.html.

[16] K. Pan, Y. Guan, J. P. Watson, and J. Wang. Strengthened MILP formulation for certain gas turbine unit commitment problems. *IEEE Transactions on Power Systems*, 31(2):1440–1448, 2016.

[17] Maurice Queyranne and Laurence A Wolsey. Tight MIP formulations for bounded up/down times and interval-dependent start-ups. *Mathematical Programming*, 164(1):129–155, 2017.

[18] D. Rajan and S. Takriti. Minimum up/down polytopes of the unit commitment problem with start-up costs. *IBM Research Report*, 2005.

[19] Cécile Rottner. *Combinatorial aspects of the unit commitment problem.* PhD thesis, Sorbonne Université, 2018.

[20] Clemens Zeile, Nicolò Robuschi, and Sebastian Sager. Mixed-integer optimal control under minimum dwell time constraints. *Mathematical Programming*, 188(2):653–694, 2021.